Revision

Example (Algebra)

Suppose $z = 4 + i$, $w = 3 - 4i$. Find

(a) $z + w$

(b) $z - w$

(c) zw

(d) $\frac{z}{w}$

(e) z^*

(f) $|z|$ and $|w|$

(g) $\arg(z)$ and $\arg(w)$

Example

Solve $z^2 + (-4-3i)z + (1+7i) = 0$

$$\begin{aligned} && 0 &= z^2 + (-4-3i)z + (1+7i) \\ \Rightarrow && z &= \frac{4+3i \pm \sqrt{(-4-3i)^2 - 4\cdot 1\cdot(1+7i)}}{2} \\ && &= \frac{4+3i \pm \sqrt{7+24i-4-28}}{2} \\ && &= \frac{4+3i \pm \sqrt{3-4i}}{2} \\ && &= \frac{4+3i \pm (2-i)}{2} \\ && &= 1+2i, 3+i \end{aligned}$$

To calculate $\sqrt{3-4i}$, write it as $a+bi$ and note that

$$\begin{aligned} && 3-4i &= (a+bi)^2 \\ && &= a^2 - b^2 + 2abi \\ \Rightarrow && a^2 - b^2 &= 3 \\ && ab &= -2 \\ \Rightarrow && 3 &= a^2 - \left(\frac{(-2)}{a}\right)^2 \\ \Rightarrow && 0 &= a^4 - 3a^2 - 4 \\ \Rightarrow && 0 &= (a^2-4)(a^2+1) \\ \Rightarrow && a &= \pm 2 \\ \Rightarrow && b &= \mp 1 \end{aligned}$$

Complex Geometry

Example

Show that $|z+w| \leq |z| + |w|$

Example

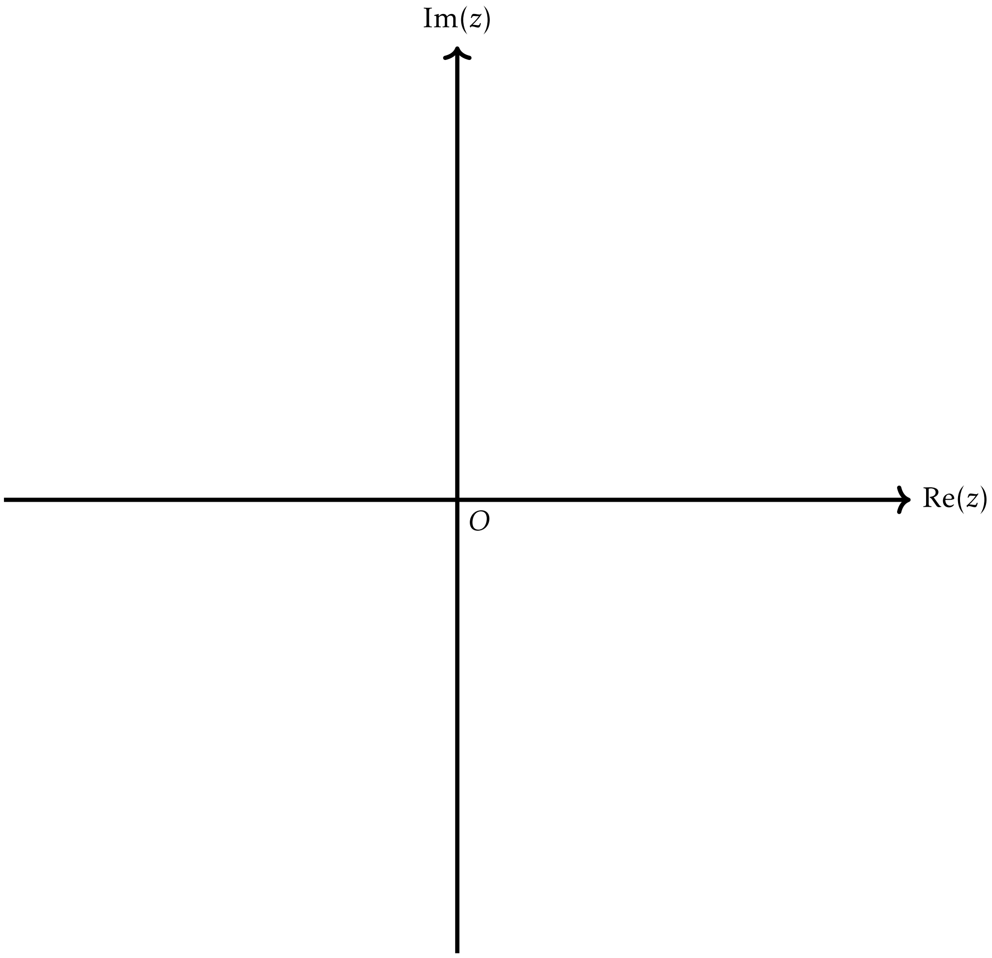
Sketch

(a) $|z-(1+i)| = 2$

(b) $\arg(z-1) = \frac{\pi}{3}$

(c) $\mathrm{Re}(z) = 3$

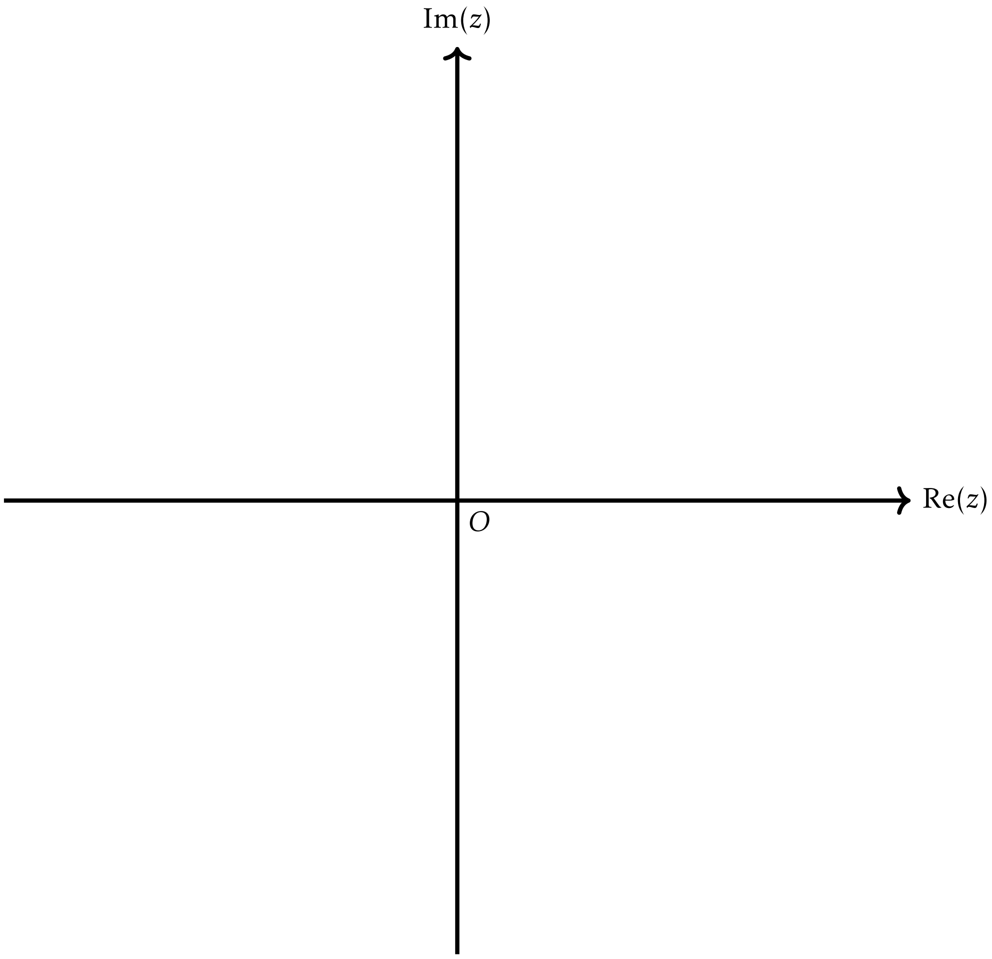
(d) $|z-1| = |z-i|$



Example

Sketch

(a) $|z-1| = 2|z-i|$



Euler's Formula

Example

Calculate e^{ix} using the Maclaurin series

$$\begin{aligned} e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots \\ &= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

Example

What are:

(a) e^0

(b) $e^{i\pi}$

(c) $e^{i\frac{\pi}{6}}$

(d) i^i

Definition. A complex number is in **exponential form** when it is written as

$$z = re^{i\theta}, r \geq 0$$

Fact — We can now write complex numbers in 4 ways:

$$z = x + iy = re^{i\theta} = r\operatorname{cis}\theta = (r;\theta)$$

Example

Write $1 + \sqrt{3}i$ in exponential form.

Example

Write $5e^{i\frac{5\pi}{6}}$ in cartesian form.

Example

Suppose $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$ then calculate

(a) z_1z_2

(b) $\frac{z_1}{z_2}$

(c) z_1^n

$$\begin{aligned} z_1z_2 &= (r_1e^{i\theta_1})(r_2e^{i\theta_2}) \\ &= r_1r_2e^{i(\theta_1+\theta_2)} \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} \\ &= \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \end{aligned}$$

$$\begin{aligned} z_1^n &= (r_1e^{i\theta_1})^n \\ &= r_1^ne^{i(n\theta_1)} \end{aligned}$$

Example (OCR Jan 2009 - FP3 Q2)

(i) Express $\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$. **[3]**

(ii) Hence find the smallest positive value of n for which $\left(\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. [2]

de Moivre's Theorem

Fact —

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Example

Write $\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^9$ in the form $a + bi$

Example

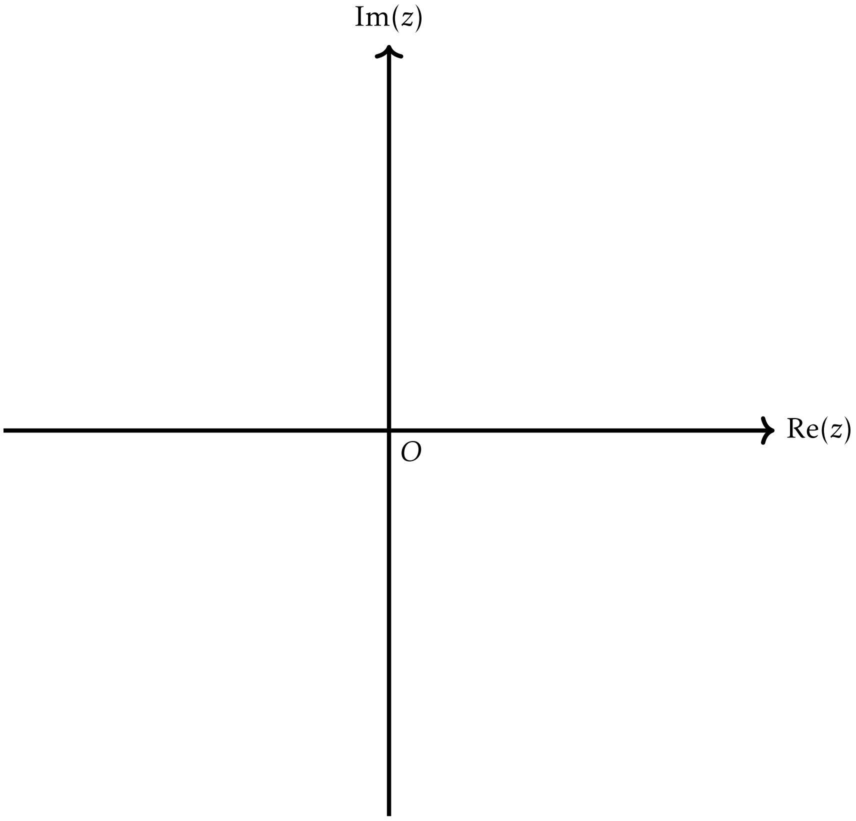
Compute $\sqrt[3]{1+i}$

$$\begin{aligned} 1+i &= \sqrt{2}e^{i\frac{\pi}{4}} \\ \Rightarrow \qquad \sqrt[3]{1+i} &= \sqrt[6]{2}e^{i\frac{\pi}{12}} \end{aligned}$$

Notice that $1+i$ *also equals* $\sqrt{2}e^{i(\frac{\pi}{4}+2n\pi)}$ *so we have more solutions, namely* $z = \sqrt[6]{2}e^{i\frac{\pi}{12}}, \sqrt[6]{2}e^{i(\frac{\pi}{12}+\frac{2\pi}{3})}, \sqrt[6]{2}e^{i(\frac{\pi}{12}+\frac{4\pi}{3})}$

Example

Plot these values on an Argand diagram.



Example

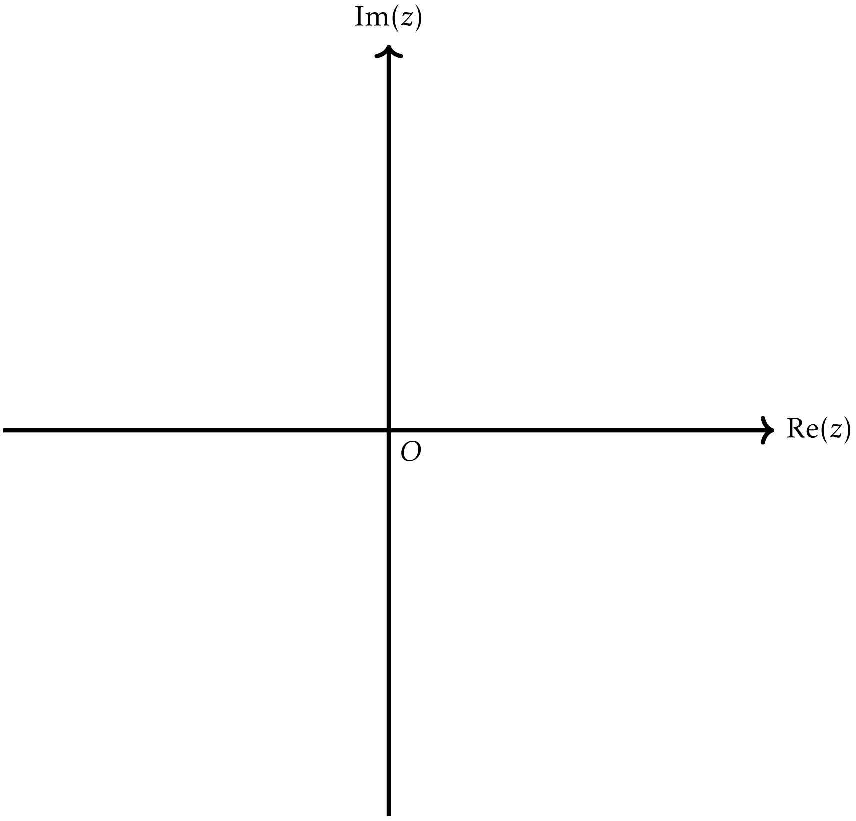
Solve $z^n - 1 = 0$

$$\begin{aligned} 1 &= e^{2k\pi i} \\ \Rightarrow \qquad w &= e^{\frac{2k\pi}{n} i} \\ &= 1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \dots, e^{\frac{2(n-1)\pi i}{n}} \end{aligned}$$

Notice if $t = e^{\frac{2\pi i}{n}}$ *the roots are* $t^0, t^1, \cdots, t^{n-1}$*. Also notice we can factories* $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$ *so we must also have that* $1 + w + \cdots + w^{n-1} = 0$ *for all* $w \neq 1$*.*

Example

Plot these values on an Argand diagram.



Trigonometry

Example

Express $\sin 3\theta$ in terms of $\sin\theta$.

$$\begin{aligned}\sin 3\theta &= Im(\cos 3\theta + i\sin 3\theta)\\ &= Im((\cos\theta + i\sin\theta)^3)\\ &= Im(c^3 + 3isc^2 - 3s^2c - is^3)\\ &= 3sc^2 - s^3\\ &= 3s(1-s^2) - s^3\\ &= 3s - 4s^3 = 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Example (OCR Jan 2010 FP3 Q7)

(i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$

[5]

(iii) Hence find the exact value of

$$\cos\left(\frac{1}{12}\pi\right)\cos\left(\frac{5}{12}\pi\right)\cos\left(\frac{7}{12}\pi\right)\cos\left(\frac{11}{12}\pi\right),$$

justifying your answer. **[5]**

Fact —

$$\cos\theta = \mathrm{Re}\left(e^{i\theta}\right) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \mathrm{Im}\left(e^{i\theta}\right) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Example

Express $\cos^6\theta$ in terms of cos and sin of multiples of θ

Let $z = e^{i\theta}$

$$
\begin{aligned}
\cos\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\\
\Rightarrow \qquad \cos^6\theta &= \left(\frac{z+\frac{1}{z}}{2}\right)^6\\
&= \frac{1}{2^6}\left(z^6+6z^5z^{-1}+15z^4z^{-2}+20z^3z^{-3}+15z^2z^{-4}+6zz^{-5}+z^{-6}\right)\\
&= \frac{1}{2^6}\left(z^6+6z^4+15z^2+20+15z^{-2}+6z^{-4}+z^{-6}\right)\\
&= \frac{1}{2^6}\left(z^6+z^{-6}+6(z^4+z^{-4})+15(z^2+z^{-2})+20\right)\\
&= \frac{1}{64}2\cos 6\theta+\frac{6}{64}2\cos 4\theta+\frac{15}{64}2\cos 2\theta+\frac{20}{64}
\end{aligned}
$$

Example (OCR June 2008 - FP3 Q4)

(i) By expressing $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^5\theta \equiv \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

[5]

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \le \theta \le \pi$. **[4]**

Example

Calculate

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \frac{1}{27}\sin 7\theta + \cdots$$

Osborn's Rule

Fact (Osborn's Rule) — Any trigonometric identity can be converted to a hyperbolic identity by:

(1) Replacing $\cos\theta$ with $\cosh\theta$

(2) Replacing $\sin\theta$ with $\sinh\theta$

(3) Negating any term containing a product of two sines (e.g. $\sin^2\theta$, $\sin A\sin B$, $\tan^2\theta$)

> **Remark** (Why does Osborn's rule work?). Substituting $i\theta$ into Euler's formula gives us a connection between trig and hyperbolic functions:
>
> $$\cos(i\theta) = \frac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$$
> $$\sin(i\theta) = \frac{e^{i(i\theta)} - e^{-i(i\theta)}}{2i} = \frac{e^{-\theta} - e^{\theta}}{2i} = \frac{-(e^{\theta} - e^{-\theta})}{2i} = \frac{i(e^{\theta} - e^{-\theta})}{2} = i\sinh\theta$$
>
> So if we take any trig identity and substitute $\theta \to i\theta$, every cos becomes cosh and every sin becomes $i\sinh$. When two sines are multiplied together, we get $(i\sinh)(i\sinh) = -\sinh^2$, which explains the sign change.

Example

Convert the following identities to their hyperbolic equivalents using Osborn's rule:

(a) $\cos^2\theta + \sin^2\theta = 1$

(b) $\sin 2\theta = 2\sin\theta\cos\theta$

(c) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(d) $1 + \tan^2\theta = \sec^2\theta$

(a) $\cosh^2\theta - \sinh^2\theta = 1$ *(sign change on* $\sinh^2$*)*

(b) $\sinh 2\theta = 2\sinh\theta\cosh\theta$ *(no product of two sines)*

(c) $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$ *(sign change on* $\sinh^2$*)*

(d) $1 - \tanh^2\theta = \text{sech}^2\theta$ *(sign change on* $\tanh^2 = \frac{\sinh^2}{\cosh^2}$*)*

Example

Use Osborn's rule to find a formula for $\cosh 3\theta$ in terms of $\cosh\theta$.

Starting from $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$*:*

$$\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta$$

(No sign changes needed as there are no sine terms.)

Fibonometry

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the same recurrence $X_{n+1} = X_n + X_{n-1}$, with initial conditions:

$$F_0 = 0, F_1 = 1 \qquad L_0 = 2, L_1 = 1$$

Fact (Binet's Formulas) — Let $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\psi = \frac{1-\sqrt{5}}{2}$. Then:

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \qquad L_n = \phi^n + \psi^n$$

Remark. Compare Binet's formulas with the exponential forms of sine and cosine:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

The Fibonacci numbers F_n play the role of sin, and the Lucas numbers L_n play the role of cos. This analogy, which Conway called Fibonometry, lets us translate trigonometric identities into Fibonacci/Lucas identities.

The following identities are parallel:

Trigonometric Identity	**Fibonacci/Lucas Identity**
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$	$F_{m+n} + F_{m-n} = F_m L_n$ (for n even)
$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$	$F_{m+n} - F_{m-n} = F_n L_m$ (for n even)
$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$	$L_{m+n} + L_{m-n} = L_m L_n$ (for n even)
$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$	$L_{m+n} - L_{m-n} = 5F_m F_n$ (for n even)
$\cos^2\theta - \sin^2\theta = \cos 2\theta$	$L_n^2 - 5F_n^2 = 4(-1)^n$
$\sin 2\theta = 2\sin\theta\cos\theta$	$F_{2n} = F_n L_n$

Remark. When n is odd, the + and − signs swap in the Fibonacci/Lucas identities (due to ψ^n being negative for odd n).

Example

Verify that $F_{2n} = F_n L_n$ using Binet's formulas.

$$\begin{aligned} F_n L_n &= \frac{\phi^n - \psi^n}{\sqrt{5}} \cdot (\phi^n + \psi^n) \\ &= \frac{\phi^{2n} - \psi^{2n}}{\sqrt{5}} \\ &= F_{2n} \end{aligned}$$

This mirrors $\sin\theta \cdot 2\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$.

Example

Use Fibonometry to prove that $L_n^2 - 5F_n^2 = 4(-1)^n$.

$$\begin{aligned}
L_n^2 - 5F_n^2 &= (\phi^n + \psi^n)^2 - 5\left(\frac{\phi^n - \psi^n}{\sqrt{5}}\right)^2 \\
&= \phi^{2n} + 2\phi^n\psi^n + \psi^{2n} - (\phi^{2n} - 2\phi^n\psi^n + \psi^{2n}) \\
&= 4\phi^n\psi^n \\
&= 4(\phi\psi)^n = 4(-1)^n
\end{aligned}$$

since $\phi\psi = \frac{(1+\sqrt{5})(1-\sqrt{5})}{4} = \frac{1-5}{4} = -1$.

This mirrors $\cos^2\theta - \sin^2\theta = \cos 2\theta$ *(a relationship between squares).*